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GROMOV–WITTEN THEORY OF ORBIFOLD PROJECTIVE LINES AND INTEGRABLE HIERARCHIES

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ABSTRACT. This lecture is a preliminary announcement of the results from my joint project with H.-H. Tseng and Y. Shen. We prove that the generating function of GW invariants for certain class of orbifolds is a tau-function for the Kac–Wakimoto hierarchy corresponding to certain conjugacy class of the Weyl group. In fact our project suggests that the Kac–Wakimoto hierarchies should admit an extension similar to the well known Extended Toda hierarchy, which governs the GW theory of $\mathbb{P}^1$.

1. INTRODUCTION

Let $X = \mathbb{P}^1_{r_1,r_2,r_3}$ be the projective line equipped with an orbifold structure, such that there are exactly 3 orbifold points of type $\mathbb{C}/(\mathbb{Z}/r_i\mathbb{Z})$, where the weights r_i are positive integers satisfying:

$$\chi := \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - 1 > 0.$$

Let us recall that the orbifold cohomology $H^*_{\rm orb}(X;\mathbb{C})$ is by definition the cohomology of its *inertia orbifold.* The latter is defined as follows. Locally, X is modelled by U/G, where U is a coordinate chart and G is a finite group acting on U. The orbifolds

$$U^g/C(g), \quad U^g = \{x \in X \ : \ g\cdot x = x\}, \quad C(g) = \{h \in G \ : \ hg = gh\}$$

parametrized by the conjugacy classes (g) of G are called *twisted sectors.* They can be glued accordingly and give rise to a disjoint union of orbifolds, called the inertia orbifold of X. Note that if we pick consistently $(g) = \{1\}$; then the resulting twisted sector is just the orbifold X.

All integers r_i satisfying the above condition are naturally in a 1-to-1 correspondance with the Dynkin diagrams of ADE type together with a choice of a *central node*: a node that splits the Dynkin diagram into 3 diagrams, which we call legs, of type A_{r_i-1} $(1 \le i \le 3)$. We will always associate such a Dynkin diagram with the orbifold X, as well as the corresponding root system and Weyl group. Furthermore, for each leg of the Dynkin diagram, we take the composition of the reflections corresponding to the nodes on the leg (in order starting from the node incident to the central node). Let us denote by σ the composition of the transformations corresponding to the legs (note that the order is irrelevant since the leg-transformations commute.

We fix a basis of the orbifold quantum cohomology $H^*_{\rm orb}(X;\mathbb{C})$ as follows:

$$\phi_{0,1}=1,\quad \phi_{0,2}=P$$

are the unit and the hyperplane class respectively and

$$\phi_{i',i''},\quad 1\le i'\le 3,\quad 1\le i''\le r_{i'}-1.$$

are the units of the corresponding twisted sectors of X. Note that the dimension of the cohomology is

$$N=r_1+r_2+r_3-1.$$

It is convenient to denote by ι the set of all pairs (i',i'') that we used above to label the basis of the cohomology. Let us denote by

$$\mathcal{D}_X(\hbar;\mathbf{t})=\exp\Big(\sum\frac{\hbar^{g-1}Q^d}{n!}\langle\tau_{k_1}(\phi_{i_1},\dots,\tau_{k_n}(\phi_{i_n}\rangle_{g,n,d}t_{k_1}^{i_1}\cdots t_{k_n}^{i_n}\Big)$$

the generating function of orbifold Gromov–Witten invariants of X. In the above formula, $\hbar$ and Q are formal parameters that keep track respectively of the genus and the degree of the holomorphic curves in X. Furthermore,

$$\mathbf{t}=(t_0,t_1,\dots),\quad t_k=(t_k^i)_{i\in\iota}$$

is a sequence of formal vector variables that we use to keep track of the various incidence (upper index i) and tangency (lower index k) constraints. Finally, using the so called *dilaton shift*

$$t_k^i=q_k^i,\quad (k,i)\ne(1,(0,1)),\quad t_1^{01}=q_1^{01}+1,$$

we identify $\mathcal{D}_X$ with a vector in the Fock space

$$\mathbb{C}_\hbar[[q_0,q_1+1,q_2,\dots]],\quad \mathbb{C}_\hbar=\mathbb{C}((\sqrt{\hbar})). \tag{1}$$

Let us recall also that for each Dynkin diagram of type ADE and a choice of a conjugacy class C in the corresponding Weyl group, Kac–Wakimoto (see [9]) constructed an integrable hierarchy, called the *Kac–Wakimoto hierarchy* (corresponding to the conjugacy class C. Our main statement can be formulated as follows: *the generating function $\mathcal{D}_X$ is a tau-function of the Kac–Wakimoto hierarchy corresponding to* $C=[\sigma]$.

The goal in this lecture is to make this statement more precise as well as to outline the main steps in the proof.

2. Frobenius manifolds and root systems

Recall that the quantum cup product is a family of multiplications $\bullet_t$ in the cohomology space $H^*(X;\mathbb{C})$ parametrized by $t\in H^*(X;\mathbb{C})$ and defined by the following genus-0 GW invariants:

$$(\phi_i\bullet_t\phi_j,\phi_k)=\sum\frac{Q^d}{n!}\,\langle\phi_i,\phi_j,\phi_k,t,\dots,t\rangle_{0,3+n,d},$$

where (,) is the intersection pairing. The quantum cup product and the pairing (,) induce on $H^*_{\rm orb}$ a Frobenius structure of conformal dimension 1 with respect to the *Euler* vector field:

$$E = \sum_{i\in\iota}(1-\deg_{\mathbb{C}}\phi_i)t^i\frac{\partial}{\partial t_i} + \Big(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-1\Big)\frac{\partial}{\partial t_{0,2}},$$

where $(t^i)_{i\in\iota}$ are the linear coordinates on M corresponding to the basis $\{\phi_i\}$ chosen above. More precisely, let $M = H^*_{\rm orb}(X;\mathbb{C})$. We consider on $M\times\mathbb{C}$ the trivial vector bundle $\mathfrak{H}$ with fiber $\mathfrak{h} := H^*_{\rm orb}(X;\mathbb{C})$. Using the linear structure on M, we can identify $\mathfrak{H}$ with the pullback of TM via the projection map $pr_1 : M\times\mathbb{C}\to M$. In particular, we frequently identify vector fields on M with sections of $\mathfrak{H}$. These structures are integrable in the sense that the following connection on $\mathfrak{H}$ is flat:

$$\nabla = d - z^{-1}\sum_{i\in\iota}(\phi_i\bullet_t)dt^i + (z^{-2}E\bullet_t - z^{-1}\mathcal{V})\,dz,$$

where

$$\mathcal{V}:\mathfrak{h}\to\mathfrak{h},\quad \mathcal{V}(\phi_i) = \Big(\frac{1}{2}-\deg_{\mathbb{C}}\phi_i\Big)\,\phi_i$$

is the so called *Hodge grading operator.*

2.1. **Periods.** For our purposes, it is more convenient to work with a family of connections that are Laplace transform of ∇. Namely, for each integer n we introduce the so called *second structure connections* ([2, 10]), on $\mathfrak{H}$:

$$\nabla^{(n)} = d + \sum_{i\in\iota}\frac{\phi_i\bullet}{(\lambda-E\bullet)}(\mathcal{V}-n-1/2)dt_i - \frac{1}{(\lambda-E\bullet)}(\mathcal{V}-n-1/2)\,d\lambda.$$

Note that $\nabla^{(n)}$ has poles along the *discriminant* locus:

$$\det(\lambda - E\bullet) = 0.$$

Let us denote by $(M\times\mathbb{C})'$ the complement to the discriminant. There exists a *calibration* operator:

$$S_t(z) = 1 + S_1(t)z^{-1} + S_2(t)z^{-2}\cdots,\quad S_k\in\operatorname{End}(\mathfrak{h}),$$

such that in a neighborhood of $z=\infty$ we have

$$S^{-1}\nabla\, S = d + \Big(z^{-2}\rho - z^{-1}\mathcal{V}\Big)dz,$$

where ρ is the cup product multiplication by $c_1(TX) = \chi P$. It follows that the differential operator

$$1 + S_1(t)(-\partial_\lambda) + S_2(t)(-\partial_\lambda)^2 + \cdots$$

provides a gauge equivalence between the connection $\nabla^{(n)}$ and the following connection:

$$d - \frac{1}{(\lambda-\rho)}(\mathcal{V}-n-1/2)\,d\lambda. \qquad (2)$$

In other words, every horizontal section of $\nabla^{(n)}$ has the form

$$I^{(n)}(t,\lambda) = \Big(1 + S_1(t)(-\partial_\lambda) + S_2(t)(-\partial_\lambda)^2 + \cdots\Big) f(\lambda), \tag{3}$$

where $f : (M \times \mathbb{C})' \to \mathfrak{h}$ is a horizontal section of (2).

Let us assume that the point $(t,\lambda) = (0,1)$ is not on the discriminant, so that we can choose it as a reference point on $M \times \mathbb{C}$. Furthermore, we fix a ray $\mathfrak{R}$ in $\{0\} \times \mathbb{C}$ starting at the reference point and approaching $\lambda = \infty$, so that we can construct uniquely a horizontal section $f(\lambda)$ of (2) for a given initial condition $f(1) \in \mathfrak{h}$. For each $a \in \mathfrak{h}$ let $I_a^{(n)}$ be the horizontal section $\nabla^{(n)}$ satisfying the initial condition:

$$\text{S-lim}_{(t,\lambda)\to(0,1)}\, I_a^{(n)}(t,\lambda) = a.$$

The S-lim here should be understood as follows. First we analytically continue the period along some path approaching $\lambda = \infty$ while keeping t fixed. We stop at $\lambda_0 \in \mathfrak{R}$ sufficiently large, so that the expansion (3) is convergent for all $|\lambda| > |\lambda_0|$. The S-lim is by definition set to be $f(1)$. Note that in this definition the branch of $I_a^{(n)}(t,\lambda)$ is specified by some path consisting of 3 pieces: from (t,λ) to (t,λ_0), a straight segment from (t,λ_0) to $(0,\lambda_0)$, and a line segment along the ray $\mathfrak{R}$. Clearly, the definition of the limit depends only on the homotopy class of the path. Finally, every other path connecting the reference point and (t,λ) is homotopic to a path of this special type. Therefore, the S-limit is well defined for any branch of a horizontal section of $\nabla^{(n)}$.

2.2. **Root systems.** The quantum cohomology is known to be semi-simple (see [11]), which means that there are local coordinates $u_i(t)$, $i \in \iota$, called *canonical*, such that

$$\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_j}, \quad i,j \in \iota.$$

If we put

$$e_i = \sqrt{\Delta_i}\frac{\partial}{\partial u_i}, \quad \Delta_i = \frac{1}{(\partial/\partial u_i\,,\,\partial/\partial u_i)}\ ;$$

then

$$e_i \bullet_t e_j = \sqrt{\Delta_i}\delta_{ij}e_j, \quad (e_i, e_j) = \delta_{ij}.$$

Note that for each $t \in M$, s.t. the quantum multiplication is semi-simple, the canonical coordinates $u_i(t)$ must be eigen-values of the operator $E\bullet_t$. Moreover, near $\lambda = u^i$ the connection $\nabla^{(n)}$ has precisely 1 section anti-invariant with respect to the local monodromy. It has the following expansion:

$$I^{(n)}(t,\lambda) = c_n\,(\lambda - u_i)^{-\frac{1}{2}-n}\Big(e_i + \cdots\Big), \tag{4}$$

where c_n is some constant depending only on n and the dots stand for some function holomorphic at $\lambda = u_i$. Let us denote by $\Delta^{(n)}$ the set of all $a \in \mathfrak{h}$ such that there exists a path in $(M \times \mathbb{C})'$ terminating at a generic point on the discriminant, such that the period $I_\alpha^{(n)}$ has an expansion of the form (4).

If $I_a^{(n)}(t,\lambda)$ is a horizontal section of $\nabla^{(n)}$; then $\partial_\lambda I^{(n)}(t,\lambda)$ is a horizontal section of $\nabla^{(n+1)}$. Therefore it coincides with $I_b^{(n+1)}(t,\lambda)$ for some $b\in\mathfrak{h}$. It is not hard to see that

$$b=(1-\rho)^{-1}(\mathcal{V}-n-1/2)\,a.$$

In other words, we have natural maps

$$r_n:\Delta^{(n)}\to\Delta^{(n+1)},\quad r_n(a):=(1-\rho)^{-1}(\mathcal{V}-n-1/2)\,a.$$

Let us denote for brevity by Δ and $\mathring{\Delta}$ the sets $\Delta^{(n)}$ respectively for $n=-1$ and $n=0$. Also, we extend $r:=r_{-1}$ by linearity to a map $r:\mathfrak{h}\to\mathfrak{h}$ and put $\mathring{\mathfrak{h}}:=r(\mathfrak{h})$.

It is easy to check that the following formula

$$(a|b)^{\sim}\;:=(I_a^{(0)}(t,\lambda),(\lambda-E\bullet_t)I_b^{(0)}(t,\lambda))=(a,(1-\rho)b)$$

gives a well defined non-degenerate pairing on $\mathfrak{h}$, namely using that $I^{(0)}$ is a horizontal section, one checks immediately that the RHS is independent of t and λ. Note that the second equality follows by taking the S-limit. Following [2], we will refer to $(\ |\ )^{\sim}$ as the *intersection pairing.* The pull back of the intersection pairing via r gives a new pairing on $\mathfrak{h}$, which we denote by $(\,|\,)$; then

Proposition 1. *The following statements hold:*

(a) *The set $\mathring{\Delta}$ is a finite root system in $(\mathring{\mathfrak{h}},(\ |\ )^{\sim})$.*
(b) *The set Δ is an affine root system in $(\mathfrak{h},(\ |\ ))$.*

Moreover, $\mathring{\Delta}$ coincides with the root system that we associated to the orbifold X and the analytical continuation along a big loop around the discriminant induces the automorphism σ of $\mathring{\Delta}$. Let us denote by $\mathring{\Lambda}$ the root lattice of $\mathring{\Delta}$.

Lemma 2. *Let $\pi_0:\mathring{\mathfrak{h}}\to\mathring{\mathfrak{h}}^{\sigma}$ be the orthogonal projection on the subspace of vectors fixed by σ; then $\pi_0(\mathring{\Lambda})=\mathbb{Z}\,\omega$, where $\omega=1+\chi P$.*

The affine root system can be described in terms of $\mathring{\Delta}$ as follows. If we delete the central node of the Dynkin diagram we obtain 3 Dynkin diagrams (the empty diagram is allowed) D_a of type A_{r_a-1}, $a=1,2,3$. Put

$$\rho^{\perp}=\sum_{i\neq k}\frac{1}{r_{a(i)}}\,\omega_i,$$

where $a(i)\in\{1,2,3\}$ is such that the simple root α_i belongs to the Dynkin diagram $D_{a(i)}$. Finally let us fix a lift $\widetilde{\alpha}\in\Delta$ of $\alpha\in\mathring{\Delta}$ for all α.

Lemma 3. *The affine root system Δ consists of the following vectors:*

$$\widetilde{\alpha}-\chi^{-1}(\omega|\alpha)^{\sim}\,\log Q\,P+2\pi\sqrt{-1}(n+(\rho^{\perp}|\alpha)^{\sim})\,P,$$

where $\alpha\in\mathring{\Delta}$, $\quad n\in\mathbb{Z}$.

2.3. **Construction of an integrable hierarchy in the Hirota form.** Following [6] and [4] we construct an integrable hierarchy in the Hirota form such that the total descendant potential $\mathcal{D}_X$ is a solution.

Given $\beta \in \Delta$ put

$$\mathbf{f}^{\beta}(t,\lambda,z) = \sum_{k\in\mathbb{Z}} \partial_\lambda^{k+1} I_\beta^{(-1)}(t,\lambda)(-z)^k,$$

where for $k < -1$ the operation ∂_λ^{k+1} is taking the flat anti-derivative. This operation is unambiguous because the operator $\mathcal{V} - 1/2 - k - 1$ is invertible for $k < -1$. The vertex operator is by definition the following element of the Heisenberg group acting on the Fock space (1):

$$\Gamma_t^{\beta}(\lambda) = e^{(\mathbf{f}^{\beta}(t,\lambda,z)_+)^{\wedge}} e^{(\mathbf{f}^{\beta}(t,\lambda,z)_-)^{\wedge}}$$

Let us denote the S-limit of the vertex operator by

$$\Gamma^{\beta}(\lambda) := \text{S-lim}\, \Gamma_t^{\beta}(\lambda), \quad \beta \in \Delta.$$

Furthermore, let us choose a set Δ' of affine roots such that the map $r : \Delta \to \mathring{\Delta}$ induces 1-to-1 correspondence between Δ' and $\mathring{\Delta}$. We define the operator $\Omega_{\Delta'}$ by the following formula:

$$\text{Res}\, \frac{d\lambda}{\lambda}\Big(\sum_{\beta\in\Delta'} b_\beta(\lambda)\Gamma^{\beta}(\lambda) \otimes_{\mathfrak{a}} \Gamma^{-\beta}(\lambda)\Big) - \Big(|\rho_\sigma^\vee|^2/|\sigma|^2 + \frac{\chi}{2\hbar}(q_0^{01}\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}} q_0^{01})^2 + \\ + \sum_{i,l}\Big(\frac{m_i}{|\sigma|} + l\Big)(q_l^i \otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}} q_l^i)(\partial_{q_l^i}\otimes_{\mathfrak{a}} 1 - 1 \otimes_{\mathfrak{a}} \partial_{q_l^i})\Big).$$

Here the notations are as follows:

$$\mathfrak{a} = \mathbb{C}_\hbar[[q_1^{01}, q_2^{01}, \dots]]$$

and the coefficients $b_\beta(\lambda)$ are defined by the following limit:

$$b_\beta^{-1}(\lambda) = \lim_{\mu\to\lambda} \Big(1 - \frac{\mu}{\lambda}\Big)^2 B_\beta(\lambda,\mu), \tag{5}$$

where $B_\beta(\lambda,\mu)$ is the phase factor from the composition of the following two vertex operators:

$$\Gamma^{\beta}(\lambda)\Gamma^{-\beta}(\mu) = B_\beta(\lambda,\mu)\ :\Gamma^{\beta}(\lambda)\Gamma^{-\beta}(\mu): .$$

Vector $\rho_\sigma^\vee \in \mathring{\mathfrak{g}}$ ($\mathring{\mathfrak{g}}$ is the simple Lie algebra corresponding to the root system $\mathring{\Delta}$) will be defined below and $|\sigma|$ is the order of the automorphism σ. In fact, the consistency of the hierarchy forces:

$$|\rho_\sigma^\vee|^2/|\sigma|^2 = \text{Res}\, \frac{d\lambda}{\lambda}\Big(\sum_{\beta\in\Delta'} b_\beta(\lambda)\Big).$$

Theorem 4. *The total descendant potential satisfies the following Hirota bi-linear equations:*

$$\Omega_{\Delta'}\ (\mathcal{D}_X \otimes \mathcal{D}_X) = 0.$$

Our goal now is to identify the Hirota equations appearing in the above theorem with the Hirota equations of the Kac-Wakimoto hierarchies.

3. Realization of the basic representation via periods

Let $\mathring{\mathfrak{g}}$ be a simple Lie algebra equipped with an invariant bi-linear pairing $(\ |\ )^{\sim}$. By definition, the affine Kac–Moody algebra corresponding to $\mathring{\mathfrak{g}}$ is the vector space

$$\mathfrak{g} := \mathring{\mathfrak{g}}[t,t^{-1}] \oplus \mathbb{C}\, K \oplus \mathbb{C}\, d$$

equipped with a Lie bracket defined by the following relations:

$$[X\ t^n, Y\ t^m] := [X,Y]\ t^{n+m} + n\delta_{n,-m}(X\,|\,Y)^{\sim}K,$$
$$[d, X\ t^n] := n(X\ t^n), \quad [K,\mathfrak{g}] := 0,$$

where $X,Y \in \mathring{\mathfrak{g}}$.

Given a finite order automorphism σ of $\mathring{\mathfrak{g}}$, Kac–Peterson (see [8]) constructed a vertex operator representation of $\mathfrak{g}$, which can be used to define an integrable hierarchy in the Hirota form. Our goal is to recall a particular case of the Kac–Peterson's construction which can be compared with the vertex operators that we introduced in the previous section.

3.1. **Simple Lie algebras.** Given a simple root system $\mathring{\Delta}$ of ADE type in some vector space $(\mathring{\mathfrak{h}}, (\ |\ )^{\sim})$ we can construct the corresponding simple Lie algebra

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{h}} \bigoplus \oplus_{\alpha\in\mathring{\Delta}} \mathring{\mathfrak{g}}_\alpha, \quad \mathring{\mathfrak{g}}_\alpha = \mathbb{C}\, A_\alpha$$

as follows. We may assume that $(\ \ )^{\sim}$ is normalized in such a way that the length of each root is $\sqrt{2}$. Identifying the root system $\mathring{\Delta}$ with the space of vanishing cycles of a simple singularity and the Weyl group $\mathring{W}$ with the monodromy group of the singularity (see [1]) we define the following co-cycle:

$$\epsilon : \mathring{\Lambda} \times \mathring{\Lambda} \to \{\pm 1\}, \quad \epsilon(\alpha,\beta) = (-1)^{\mathring{\mathrm{SF}}(\alpha,\beta)},$$

where $\mathring{\mathrm{SF}}$ is the Seifert form (linking number between the vanishing cycles). Using the standard properties of the Seifert form (see [1]) we get that this cocycle is $\mathring{W}$-invariant and it satisfies the following properties:

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)^{\sim}}, \quad \epsilon(\alpha,\alpha) = (-1)^{|\alpha|^2/2},$$

where $|\alpha|^2 := (\alpha|\alpha)^{\sim}$. Once we have such a co-cycle we can recall the so called Frenkel–Kac construction [5], and define a Lie bracket on $\mathring{\mathfrak{g}}$ as follows:

$$[A_\alpha, A_{-\alpha}] = \epsilon(\alpha,-\alpha)\alpha$$
$$[A_\alpha, A_\beta] = \epsilon(\alpha,\beta)A_{\alpha+\beta}, \quad \text{if} \quad (\alpha|\beta)^{\sim} = -1$$
$$[A_\alpha, A_\beta] = 0, \quad \text{if} \quad (\alpha|\beta)^{\sim} \geq 0.$$

Moreover, given an element $\sigma \in \mathring{W}$ we extend σ to a Lie algebra automorphism $\sigma : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$ by setting $\sigma(A_\alpha) = A_{\sigma(\alpha)}$. Moreover, we can extend the bilinear pairing $(\ |\ )^{\sim}$ to $\mathring{\mathfrak{g}}$ in the following way:

$$(A_\alpha|A_{-\alpha})^{\sim} := \epsilon(\alpha,-\alpha), \quad (A_\alpha|A_\beta)^{\sim} := (A_\alpha|H)^{\sim} = 0, \quad \forall \beta \neq -\alpha,\ H \in \mathring{\mathfrak{h}}.$$

The extended pairing is both $\mathring{\mathfrak{g}}$-invariant (with respect to the adjoint representation) and $\mathring{W}$-invariant.

3.2. **Twisted realization of the affine Lie algebra.** Put $\eta = e^{2\pi\sqrt{-1}/|\sigma|}$; then we extend further the action of σ to the affine Lie algebra $\mathfrak{g}$ by

$$\sigma \cdot (X \otimes t^n) = \sigma(X) \otimes (\eta^{-1}t)^n, \quad \sigma \cdot K = K, \quad \sigma \cdot d = d.$$

Let $\mathfrak{g}^\sigma$ be the Lie subalgebra of σ-fixed points. According to Kac (see [7], Theorem 8.6.) $\mathfrak{g}^\sigma \cong \mathfrak{g}$. Let us recall the isomorphism. The fixed points subspace $\mathring{\mathfrak{g}}^\sigma$ contains a Cartan subalgebra $\mathring{\mathfrak{g}}_0$. We have a corresponding decomposition into root subspaces

$$\mathring{\mathfrak{g}} = \bigoplus_{\alpha' \in \mathring{\Delta}'} \mathring{\mathfrak{g}}_{\alpha'},$$

where $\mathring{\Delta}' \subset \mathring{\mathfrak{g}}_0$ are the corresponding roots. Note that since the root subspaces are 1 dimensional, they must be eigen-subspaces of σ. Therefore, by choosing a set of simple roots α_i', $i = 1, 2, \ldots, N-1$ in $\mathring{\Delta}'$ we can uniquely define an integral vector $s = (s_1, \ldots, s_{N-1})$, $0 \le s_i < |\sigma|$ s.t., the eigenvalue of the eigensubspace $\mathring{\mathfrak{g}}_{\alpha_i'}$ is η^{s_i}. Put

$$\rho_\sigma : \mathring{\mathfrak{g}}_0 \to \mathring{\mathfrak{g}}_0, \quad \rho_\sigma = \sum_{i=1}^{N-1} s_i \omega_i',$$

where $\omega_i'(1 \le i \le N-1)$ are the fundamental weights corresponding to the simple roots $\alpha_i'(1 \le i \le N-1)$; then it is easy to see that an isomorphism

$$\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^\sigma$$

is given by the following map

$$\Phi(Xt^n) = t^{n|\sigma| + \mathrm{ad}_{\rho_\sigma^\vee}} X + \delta_{n,0}\, (\rho_\sigma^\vee | X) K \tag{6}$$

$$\Phi(K) = |\sigma|\, K$$

$$\Phi(d) = |\sigma|^{-1}\Big(d - \rho_\sigma^\vee - \frac{1}{2}(\rho_\sigma^\vee|\rho_\sigma^\vee)^\sim K\Big), \tag{7}$$

where $\rho_\sigma^\vee \in \mathring{\mathfrak{g}}_0$ is the dual to ρ_σ, i.e., $(\rho_\sigma^\vee | \cdot)^\sim = \rho_\sigma(\,\cdot\,)$ and

$$t^{\mathrm{ad}_{\rho_\sigma^\vee}} X = \exp\Big(\log t \ \mathrm{ad}_{\rho_\sigma^\vee}\Big) X.$$

To see that this definition is meaningful, one has to notice first that

$$\sigma = \exp\Big(2\pi\sqrt{-1}\mathrm{ad}_{\rho_\sigma^\vee/|\sigma|}\Big).$$

It follows that the RHS is a single valued function and that the resulting loop is σ-invariant.

3.3. **Periods and vertex operators.** In this subsection we will derive explicit formulas for the S-limit of the vertex operator $\Gamma_t^{\beta}(\lambda)$. To begin with, let us denote by $*$ the natural involution in our index set that comes from Poincaré duality, i.e., the involution is uniquely defined so that the Poincaré pairing has the following symmetry:

$$(\phi_i, \phi_{j^*}) = \frac{1}{r_{i'}}\,\delta_{i,j}, \quad \forall i, j \in \iota,$$

where we set $r_0 := 1$. It will be convenient to introduce also

$$\omega_c := \omega/\chi, \quad d_c := (\omega_c|\omega_c)^{\sim} = 1/\chi.$$

It can be proved that ω_c is the fundamental weight corresponding to the central node of the Dynkin diagram (see Lemma 2). Furthermore, we put

$$\mathring{H}_j = \sqrt{|\sigma|\, r_{j'}}\,\phi_j, \quad j = (j', j'') \in \iota_{\mathrm{tw}},$$

where ι_{tw} is the index set of the cohomology classes supported on the twisted sectors, i.e., $j \in \iota$ such that $j' \neq 0$, and

$$\mathring{H}_{(0,1)} := \mathring{H}_{(0,2)} := \sqrt{|\sigma|\,\chi}\,\omega_c;$$

then it is easy to see that these vectors form a basis of eigenvectors of $\mathring{\mathfrak{h}}$ for the monodromy σ, satisfying the following orthogonality relations:

$$(\mathring{H}_i|\mathring{H}_{j^*})^{\sim} = |\sigma|\,\delta_{i,j}, \quad \forall i, j \in \iota.$$

The eigenavlue of $\mathring{H}_i$ is $e^{-2\pi\sqrt{-1}d_i}$ for $i \in \iota$. Put $d_i = 1 - m_i/|\sigma|$, $i \in \iota$; then

$$m_{(0,1)} = 0, \quad m_{(0,2)} = |\sigma|, \quad m_i = i''\,\frac{|\sigma|}{r_{i'}}, \quad i \in \iota_{\mathrm{tw}}.$$

Note that these numbers have the following symmetry:

$$m_i + m_{i^*} = |\sigma|, \quad \forall i \in \iota.$$

Let

$$\beta = r^{-1}(\alpha) - (\omega_k|\alpha)^{\sim}\,\log Q\,P + 2\pi\sqrt{-1}(n + (\rho^{\perp}|\alpha)^{\sim}) \tag{8}$$

be an affine root; then by definition we have

$$\begin{aligned} I_{\beta}^{(-1)}(\lambda) \;&=\; (\alpha|\omega_c)^{\sim}\,\lambda + (\alpha|\omega)^{\sim}(\log\lambda - C_0)\,P + 2\pi\sqrt{-1}(n + (\rho^{\perp}|\alpha)^{\sim})\,P + \\ &\quad \sum_{i\in\iota_{\mathrm{tw}}} (\alpha|H_{i^*})^{\sim}\,\sqrt{r_{j'}/|\sigma|}\,\frac{\lambda^{d_j}}{d_j}\,\phi_j, \end{aligned}$$

where $C_0 = \log Q^{d_c}$. From here we find that the remaining periods are:

$$I_{\beta}^{(l)}(\lambda) = (-1)^l l!(\alpha|\omega)^{\sim}\lambda^{-l-1}P + \sum_{i\in\iota_{\mathrm{tw}}} (\alpha|H_{i^*})^{\sim}(d_i - 1)\cdots(d_i - l)\lambda^{d_i-l-1}\sqrt{r_{i'}/|\sigma|}\phi_i,$$

for $l \geq 1$,

$$I_{\beta}^{(0)}(\lambda) = (\alpha|\omega_c)^{\sim} + (\alpha|\omega)^{\sim}\lambda^{-1}P + \sum_{i\in\iota_{\mathrm{tw}}} (\alpha|H_{i^*})^{\sim}\lambda^{d_i-1}\sqrt{r_{i'}/|\sigma|}\phi_i,$$

and

$$I_\beta^{(-1-l)}(\lambda) = (\alpha|\omega_c)^\sim \frac{\lambda^{l+1}}{(l+1)!} + (\alpha|\omega)^\sim \frac{\lambda^l}{l!}(\log\lambda - C_l)\, P + 2\pi\sqrt{-1}n\, \frac{\lambda^l}{l!}\, P + \sum_{i\in\iota_{\rm tw}} (\alpha|H_{i^*})\sqrt{r_{j'}/|\sigma|}\, \frac{\lambda^{d_j+l}}{d_j(d_j+1)\cdots(d_j+l)}\, \phi_j,$$

where $C_l (l\geq 1)$ are constants defined recursively by $C_l = C_{l-1} + 1/l$.

Applying our quantization formalism, together with the substitutions $\lambda = \zeta^{|\sigma|}/|\sigma|$ and

$$(9) \qquad y_{02,l} = \frac{1}{\sqrt{\hbar}}\, \frac{|\sigma|^{d_{02}}}{\sqrt{|\sigma|\chi}}\, \frac{q_l^{02}}{m_i(m_i+|\sigma|)\cdots(m_i+l|\sigma|)},$$

$$(10) \qquad y_{i,l} = \frac{1}{\sqrt{\hbar}}\, \frac{|\sigma|^{d_i}}{\sqrt{|\sigma| r_{i'}}}\, \frac{q_l^i}{m_i(m_i+|\sigma|)\cdots(m_i+l|\sigma|)}, \quad i\in\iota_{\rm tw},$$

we get that the vertex operator $\Gamma^\beta(\lambda)$ has the form:

$$\Gamma^\beta(\lambda) = U_\beta(\lambda)\, \Gamma_0^\beta(\lambda)\, \Gamma_*^\beta(\zeta),$$

where

$$U_\beta(\lambda) = \exp\Big(\sum_{l=1}^\infty \Big(\omega(\alpha)(\log\lambda - C_l) + 2\pi\sqrt{-1}(n+(\rho^\perp|\alpha)^\sim)\Big)\frac{\lambda^l}{l!}\, q_l^{01}/\sqrt{\hbar}\Big),$$

$$\Gamma_0^\beta(\lambda) = \exp\Big(\Big(\omega(\alpha)\, \log\frac{\lambda}{Q^{d_c}} + 2\pi\sqrt{-1}(n+(\rho^\perp|\alpha)^\sim)\Big)\frac{q_0^{01}}{\sqrt{\hbar}}\Big)\, \exp\Big(-\omega_c(\alpha)\sqrt{\hbar}\frac{\partial}{\partial q_0^{01}}\Big),$$

and

$$\Gamma_*^\beta(\zeta) = \exp\Big(\sum_{i,l}(\alpha|H_i)^\sim\, \zeta^{m_i+l|\sigma|}\, y_{i,l}\Big)\, \exp\Big(\sum_{i,l}(\alpha|H_{i^*})^\sim\, \frac{\zeta^{-m_i-l|\sigma|}}{-m_i-l|\sigma|}\, \frac{\partial}{\partial y_{i,l}}\Big),$$

where the sums are over all $i\in\iota\backslash\{(0,1)\}$ and $l\geq 0$.

3.4. **Twisted realization of the basic representation.** Following [8], we would like to recall the realization of the basic level 1 representation of the affine Lie algebra $\mathfrak{g}$ corresponding to the automorphism σ. The idea is to construct a representation of the Lie algebra $\mathfrak{g}^\sigma$. Note that the central charge K of $\mathfrak{g}^\sigma$ must act by the scalar $1/|\sigma|$.

Let us introduce the following generating series:

$$X_\alpha(\zeta) = \sum_{n\in\mathbb{Z}} A_{\alpha,n}\, \zeta^{-n} = \frac{1}{|\sigma|}\sum_{l=1}^{|\sigma|}\sum_{n\in\mathbb{Z}} \eta^{-nl}(A_{\sigma(\alpha)}t^n)\zeta^{-n}, \quad \alpha\in\mathring{\Delta}$$

and the following vectors:

$$H_{i,l} = H_i t^{m_i+l|\sigma|}, \quad i\in\iota.$$

After a direct computation we get

$$[H_{i,l}, X_\alpha(\zeta)] = (\alpha|H_i)^\sim \zeta^{m_i+l|\sigma|} X_\alpha(\zeta).$$

It follows that $X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta)$, where $E^*_\alpha(\zeta)$ is the following vertex operator:

$$\exp\Big(\sum_{i,l}(\alpha|H_i)^\sim H_{i^*,-l-1}\frac{\zeta^{m_i+l|\sigma|}}{m_i+l|\sigma|}\Big)\exp\Big(\sum_{i,l}(\alpha|H_{i^*})^\sim H_{i,l}\frac{\zeta^{-m_i-l|\sigma|}}{-m_i-l|\sigma|}\Big) \tag{11}$$

and $X^0_\alpha(\zeta)$ is an operator commuting with $H_{i,l}$ for all $i \in \iota\backslash\{(0,1)\}$ and $l \in \mathbb{Z}$.

Let $\mathfrak{S}$ be the subgroup of the affine Kac–Moody Lie group generated by the lifts of the following loops:

$$h_{\alpha,\beta} = \exp\Big(\alpha \log t^{|\sigma|} + 2\pi\sqrt{-1}\ \beta\Big),$$

where $\alpha \in \mathring{\mathfrak{h}}^\sigma$ and $\beta \in \mathfrak{h}$ are such that

$$\sigma(\beta) - \beta + \alpha \in \mathring{\Lambda}.$$

Let us point out that under the analytical continuation around $t = 0$, the loop $h_{\alpha,\beta}$ gains the factor $e^{2\pi\sqrt{-1}|\sigma|\alpha}$. The latter must be 1 because:

$$|\sigma|\alpha = (\alpha+\sigma(\beta)-\beta)+\sigma(\alpha+(\sigma(\beta)-\beta))+\cdots+\sigma^{|\sigma|-1}(\alpha+(\sigma(\beta)-\beta)\in\mathring{\Lambda}.$$

It follows that $h_{\alpha,\beta}$ is single valued and σ-invariant, i.e., it defines an element of the affine Kac–Moody loop group acting on $\mathfrak{g}^\sigma$ by conjugation. Furthermore, the operators

$$H_0,\quad H_{i,l},\quad H_{i^*,-l-1}\ (l\geq 0, i\in\iota\backslash\{(0,1)\},\quad K$$

span a *Heisenberg* Lie algebra $\mathfrak{s}$. The main result of Kac–Peterson is the following: *the basic representation of* $\mathfrak{g}^\sigma$ *remains irreducible when restricted to the pair* $(\mathfrak{s},\mathfrak{S})$.

Let $\mathfrak{s}_-$ be the Lie subalgebra of $\mathfrak{s}$ spanned by the vectors $H_{i^*,-l-1}$, $i\in\iota\backslash\{(0,1)\}$, $l\geq 0$. The basic representation can be realized on the following vector space:

$$V_x = S^*(\mathfrak{s}_-)\otimes\mathbb{C}[e^\omega]e^{x\omega},$$

where x is a complex number, the first factor in the tensor product is the symmetric algebra on $\mathfrak{s}_-$, and the second one is the group algebra of the lattice $\mathbb{Z}\,\omega = \pi_0(\mathring{\Lambda})$. We will refer to $|0\rangle := 1\otimes e^{x\omega}$ as the vacuum vector. Slightly abusing the notation we put $\partial_\omega := \frac{\partial}{\partial\omega} - x$ (so that $\partial_\omega|0\rangle = 0$). Let us represent the Heisenberg algebra $\mathfrak{s}$ on $\mathbb{C}[e^\omega]e^{x\omega}$ by letting all generators act trivially, except for $H_0\mapsto\omega(H_0)\partial_\omega$. This way V_x becomes naturally a $\mathfrak{s}$-module. Furthermore, put

$$E^0_\alpha(\zeta) = \exp\Big(\omega_c(\alpha)\omega\Big)\exp\Big((\omega(\alpha)\ \log\zeta^{|\sigma|}\ +2\pi\sqrt{-1}\ (\rho^\perp|\alpha)^\sim)\partial_\omega\Big)$$

and $E_\alpha(\zeta) = E^0_\alpha(\zeta)E^*_\alpha(\zeta)$, where $E^*_\alpha(\zeta)$ is defined by formula (11).

Theorem 5. *There are constants* c_α, $\alpha\in\mathring{\Delta}$ *such that the maps* (π_0 *was introduced in Lemma 2*)

$$\begin{aligned} X_\alpha(\zeta) &\mapsto c_\alpha\zeta^{|\sigma||\pi_0(\alpha)|^2/2}E_\alpha(\zeta),\quad \alpha\in\mathring{\Delta}\\ K &\mapsto 1/|\sigma|,\\ d &\mapsto -\frac{1}{2|\sigma|}|\rho^\vee_\sigma|^2-\frac{1}{2}H_0^2-\sum_{i,l}H_{i^*,-l-1}H_{i,l}. \end{aligned}$$

lift the representation of the Heisenberg algebra $\mathfrak{s}$ on V_x to a representation of the affine Lie algebra $\mathfrak{g}^\sigma$.

Using the commutation relations of the vertex operators we can find a formula for the products $c_\alpha c_{-\alpha}$. Let us introduce the following bi-multiplicative function on $\mathring{\Lambda}$:

$$B(\alpha,\beta) = |\sigma|^{-(\alpha|\beta)^\sim} \prod_{l=1}^{|\sigma|-1} (1-\eta^l)^{(\sigma^l(\alpha)|\beta)^\sim}.$$

Lemma 6. *The constants c_α satisfy the following identity:*

$$c_\alpha c_{-\alpha} = \frac{\epsilon(\alpha,-\alpha)}{B(\alpha,-\alpha)}\, e^{2\pi\sqrt{-1}(\rho^\perp|\alpha)^\sim(\omega_c|\alpha)^\sim}.$$

4. The Kac–Wakimoto hierarchy

Following Kac-Wakimoto, we can define an integrable hierarchy in the Hirota form whose solutions are parametrized by the orbit of the vacuum vector $|0\rangle$ of the affine Kac–Moody group. A vector $\tau \in V_x$ belongs to the orbit iff $\Omega_x(\tau\otimes\tau)=0$, where Ω_x is the operator representing the following bi-linear Casimir operator:

$$\sum_{\alpha\in\mathring{\Delta}}\sum_n \frac{1}{(A_\alpha|A_{-\alpha})} A_{\alpha,n}\otimes A_{-\alpha,-n} + K\otimes d + d\otimes K + \frac{1}{|\sigma|}H_0\otimes H_0 + \\ +\frac{1}{|\sigma|}\sum_{i,l}\Big(H_{i,l}\otimes H_{i^*,-l-1} + H_{i^*,-l-1}\otimes H_{i,l}\Big),$$

where the second sum is over all $i\in\iota\backslash\{(0,1)\}$ and all $l\geq 0$. On the other, hand we have

$$\sum_n \frac{1}{(A_\alpha|A_{-\alpha})} A_{\alpha,n}\otimes A_{-\alpha,-n} = \mathrm{Res}_{\zeta=0}\frac{d\zeta}{\zeta}a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta),$$

where

$$a_\alpha(\zeta) = B(\alpha,\alpha)\,\zeta^{|\sigma|\,|\pi_0(\alpha)|^2}\, e^{2\pi\sqrt{-1}\rho^\perp(\alpha)\omega_c(\alpha)}.$$

We identify the symmetric algebra $S^*(\mathfrak{s}_-)$ with the Fock space $\mathbb{C}[y]$, where $y=(y_{i,l})$ is a sequence of formal variables indexed by $i\in\iota\backslash\{(0,1)\}$ and $l\geq 0$, by identifying $H_{i^*,-l-1}=(m_i+l|\sigma|)y_{i,l}$; then

$$H_{i,l}=\frac{\partial}{\partial y_{i,l}},\quad H_0 = (|\sigma|\chi)^{1/2}\,\partial_\omega,\quad K=1/|\sigma|,$$

and

$$d = -\frac{|\rho_\sigma^\vee|^2}{2|\sigma|} - \frac{\chi}{2}\partial_\omega^2 - \sum_{i,l}(m_i+l|\sigma|)y_{i,l}\partial_{y_{i,l}}.$$

The Hirota equations then assume the following form. A vector τ belongs to the orbit if and only if the following bilinear equations hold:

$$\operatorname{Res}\frac{d\zeta}{\zeta}\Big(\sum_{\alpha\in\hat{\Delta}} a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\Big)\,\tau\otimes\tau = \Big(|\rho_\sigma^\vee|^2/|\sigma|^2 + \frac{\chi}{2|\sigma|}(\partial_\omega\otimes 1 - 1\otimes\partial_\omega)^2 +$$

$$+\frac{1}{|\sigma|}\sum_{i,l}(m_i + l|\sigma|)(y_{i,l}\otimes 1 - 1\otimes y_{i,l})(\partial_{y_{i,l}}\otimes 1 - 1\otimes\partial_{y_{i,l}})\Big)\tau\otimes\tau.$$

Finally, let us point out that the constant $|\rho_\sigma^\vee|^2/|\sigma|^2$ can be found from the consistency of the hierarchy. Namely if $\tau = |0\rangle$ we get that

$$|\rho_\sigma^\vee|^2/|\sigma|^2 = \sum_{\alpha:\ \omega_c(\alpha)=0} a_\alpha(\zeta).$$

We will refer to the above hierarchy as the *non-extended ADE-Toda hierarchy.* Its relation to the standard Toda lattice hierarchies will be addressed in future investigation.

Recall the change of the dynamical variables (9)–(10) and the identification $\lambda = \zeta^{|\sigma|}/|\sigma|$. In order to compare the Casimir operators $\Omega_{\Delta'}$ and Ω_x we need to introduce some kind of a discrete Fourier transform. Namely, we define a map

$$\mathcal{F}_x : \mathbb{C}_\hbar[[q_0, q_1 + 1, q_2 \ldots]] \to \mathfrak{a}\otimes_{\mathbb{C}} V_x,$$

by the following formula

$$\mathcal{F}_x(f(q_0^{01},\ldots)) = \sum_{n\in\mathbb{Z}} f((x+n)\sqrt{\hbar},\ldots)\, e^{(n+x)\omega}\Big(|\sigma|^\chi Q\Big)^{\frac{1}{2}n^2},$$

where the dots stand for the remaining $\mathbf{q}$-variables on which f depends. It is easy to check that

$$\mathcal{F}_x \circ q_0^{01}/\sqrt{\hbar} = (\partial_\omega + x)\circ\mathcal{F}_x \tag{12}$$

and

$$\mathcal{F}_x \circ e^{-m\sqrt{\hbar}\partial/\partial q_0^{01}} = e^{m\omega}\Big(|\sigma|^\chi Q\Big)^{\frac{1}{2}m^2 + m\partial_\omega}\circ\mathcal{F}_x. \tag{13}$$

Proposition 7. *The Fourier transform $\mathcal{F}_x$ intertwines the Casimir operators $\Omega_{\Delta'}$ and Ω_x, more precisely,*

$$(\mathcal{F}_x\otimes_{\mathfrak{a}}\mathcal{F}_x)\circ\Omega_{\Delta'} = \Omega_x\circ(\mathcal{F}_x\otimes_{\mathfrak{a}}\mathcal{F}_x).$$

This proposition in conjunction with Theorem 4 gives the following corollary.

Corollary 8. *The Fourier transform $\mathcal{F}_x(\mathcal{D}_X)$ is a solution to the Hirota bi-linear equations of the non-extended ADE-Toda hierarchy.*

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